Theoretical and Applied Fracture Mechanics 35 (2001) 129–135

theoretical and applied fracture mechanics

www.elsevier.com/locate/tafmec

Complementary energy variational approach for plane elastic problems with singularities

F. Fraternali *

Department of Civil Engineering, University of Salerno, 84084 Fisciano (SA), Italy

Abstract

Presented is the numerical analysis of plane elastic problems involving stress concentrations and/or singularities using a physically meaningful complementary energy variational approach. The continuum body is modeled by a non-conventional truss structure. Stress distributions in laminated composite bodies and orthotropic sheets with a through crack are obtained. The present results are compared with the analytical solutions for different numerical methods. © 2001 Elsevier Science Ltd. All rights reserved.

1. Introduction

In recent studies [1,2], a numerical method has been proposed for plane elastic problems of simplyconnected bodies. Polyhedral approximations are used for the Airy stress function over triangular meshes. Such approximations are non-conforming and give rise to singular stresses at the interfaces between the triangular elements. In particular, the "skeleton" of the mesh can be regarded as pin-joint truss structure that discretizes the continuum.

The method known as the lumped stress method (LSM) involves a relaxation of the complementary energy that enlarges the space of admissible stress functions. It considers the stress field as linear Dirac deltas. Beside the (primary) triangular mesh, a dual mesh of polygons is introduced to average the stress singularities in the neighborhood of the nodes of the primary mesh. Statically admissible approximate solutions are found by an unconstrained process of minimization. Numerical integrations are not required and singular behavior such as concentrated forces, cracks or structures composed of both two-dimensional and one-dimensional elements can be easily treated. The basic idea of the method is derived from previous studies for biharmonic problems [3] and thin plates [4].

From the physical point of view, LSM offers a rational way to approximate the behavior of a 2D continuum body using a (non-conventional) truss structure.

The present work deals with applications of the lumped stress method to plane elastic problems involving anisotropic behavior and stress concentrations or singularities. The convergence of LSM approximations is shown by comparing the results with analytical solutions and others available in the literature.

2. The lumped stress method

Consider the elastic problem of a plane elastic body Ω of polygonal boundary $\partial\Omega$. It is subjected

* Tel.: +39-89-964083; fax: +39-89-964045.
E-mail address: fraternali@diima.unisa.it (F. Fraternali).

PII: S0167-8442(00)00054-9

to body force $\mathbf{b}$ in Ω, surface tractions $\mathbf{p}$ over a portion Γ_p of $\partial\Omega$, and displacements $\bar{\mathbf{u}}$ over $\Gamma_u = \partial\Omega - \Gamma_p$. In what follows, x_1, x_2 denote the Cartesian coordinates in the plane Ω. $\mathbf{T}$ is the stress field and $\mathbf{T}^*$ is an arbitrary solution of the equilibrium equation $\operatorname{div}\mathbf{T} + \mathbf{b} = \mathbf{0}$. Note that

$$T_{\alpha\beta} - T^*_{\alpha\beta} = e_{\alpha\gamma} e_{\beta\delta} H_{\gamma\delta}(\varphi), \tag{1}$$

where φ is the Airy stress function, and

$$\mathbf{H}(\varphi) = \begin{pmatrix} \varphi_{,11} & \varphi_{,12} \\ \varphi_{,21} & \varphi_{,22} \end{pmatrix}. \tag{2}$$

The range of the Greek indices is $\{1, 2\}$ with $e_{\alpha\beta}$ being the two-dimensional alternator.

2.1. Minimization

It is easy to show from the principle of minimum complementary energy that the elastic problem at hand is equivalent to minimizing the functional

$$E(\varphi) = \frac{1}{2} \int_\Omega \mathbf{H}(\varphi) \cdot \mathscr{A}[\mathbf{H}(\varphi)] \, \mathrm{d}a - \ell(\varphi) \tag{3}$$

over the set of admissible stress functions. This set consists of those stress functions φ which belong to the Sobolev space $H^2(\Omega)$ and matches suitable boundary conditions on Γ_p with the surface tractions $\mathbf{p}^* = \mathbf{p} - \mathbf{T}^*\hat{\mathbf{n}}$ [2]. In Eq. (2), $\ell(\varphi)$ is the work of reactive stresses and $\mathscr{A}$ is the fourth-order tensor defined by

$$\mathscr{A}_{\alpha\beta\gamma\delta} = e_{\alpha\mu} e_{\beta\nu} e_{\gamma\rho} e_{\delta\sigma} A_{\mu\nu\rho\sigma}, \tag{4}$$

where $A_{\mu\nu\rho\sigma}$ are the components of the elastic compliance tensor $\mathbf{A}$.

2.2. Discretization

Now, introduce a triangulation (primary mesh)

$$\Pi_h = \{\Omega_m, \ m \in \{1, 2, \ldots, M\}\} \tag{5}$$

of the domain Ω, expanding outside Ω in correspondence with Γ_p, and having the external nodes arbitrarily close to the boundary ($\varepsilon \to 0$ in Fig. 1). Introduce also a dual mesh

$$\hat{\Pi}_h = \left\{\hat{\Omega}_n, \ n \in \{1, 2, \ldots, N\}\right\} \tag{6}$$

whose elements $\hat{\Omega}_n$ are polygons which enclose the nodes $\mathbf{x}_n$ of the primary mesh, and have edges crossing at the middle points with those of the triangles Ω_m, Fig. 1.

The lumped stress method considers a discrete set S_{ph} (h = mesh size) of polyhedral approximations $\hat{\varphi}$ of the stress function over the primary mesh Π_h. It seeks for the optimal approximation $\hat{\varphi}_h$ in S_{ph} by minimizing the modified energy functional

$$E_h(\hat{\varphi}) = \frac{1}{2} \sum_{n=1}^{N} \frac{1}{\operatorname{ar}(\hat{\Omega}_n)} \int_{\hat{\Omega}_n} \mathbf{H}(\hat{\varphi}) \, \mathrm{d}a \cdot \int_{\hat{\Omega}_n} \mathscr{A}\left[\mathbf{H}(\hat{\varphi})\right] \, \mathrm{d}a - l(\varphi). \tag{7}$$

Notice that $\mathbf{H}(\hat{\varphi})$ consists of a combination of linear Dirac deltas along the edges of the triangles Ω_m. Consider, in particular, the edge connecting the nodes $\mathbf{x}_n$ and $\mathbf{x}_s$, and denote by $\hat{P}^s_n(\hat{\varphi})$ the jump of $\nabla\hat{\varphi} \cdot \hat{\mathbf{h}}^s_n$ across this edge ($\mathbf{h}^s_n$ = unit vector orthogonal to $\mathbf{x}_s - \mathbf{x}_n$). Such a

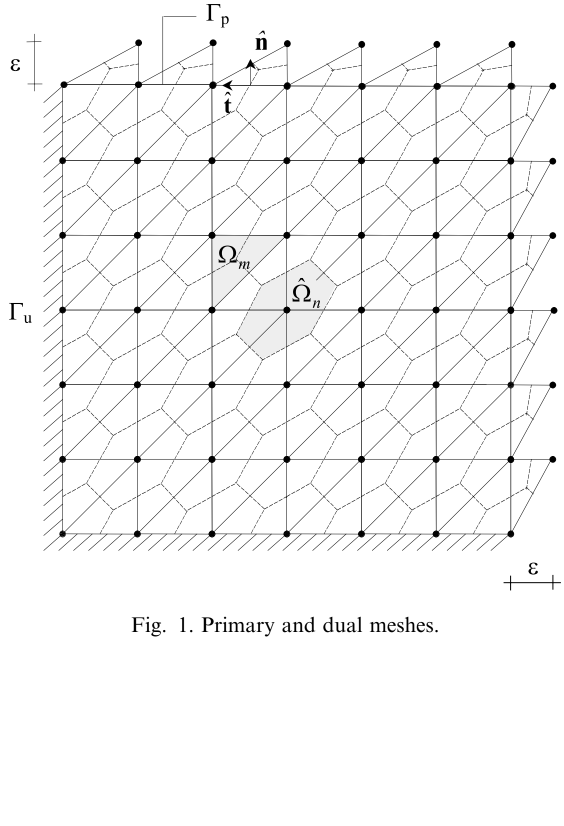

Fig. 1. Primary and dual meshes.

quantity can be regarded as the axial force carried by the bar $n - s$ of a pin-joint truss with the same geometry as the "skeleton" for the primary mesh.

Now, express the functional in Eq. (7) in terms of the quantities $\hat{P}_n^s(\hat{\varphi})$ [2]

$$E_h(\hat{\varphi}) = \frac{1}{2}\sum_{n=1}^{N}\sum_{s,t=1}^{S_n}\hat{\mathscr{A}}_n^{st}\hat{P}_n^s(\hat{\varphi})\hat{P}_n^t(\hat{\varphi}) - \sum_{n=1}^{N_u}\mathbf{R}_n(\hat{\varphi})\cdot\bar{\mathbf{u}}_n, \tag{8}$$

where S_n is the number of connections of the node $\mathbf{x}_n$, N_u the number of nodes belonging to Γ_u, $\bar{\mathbf{u}}_n$ the

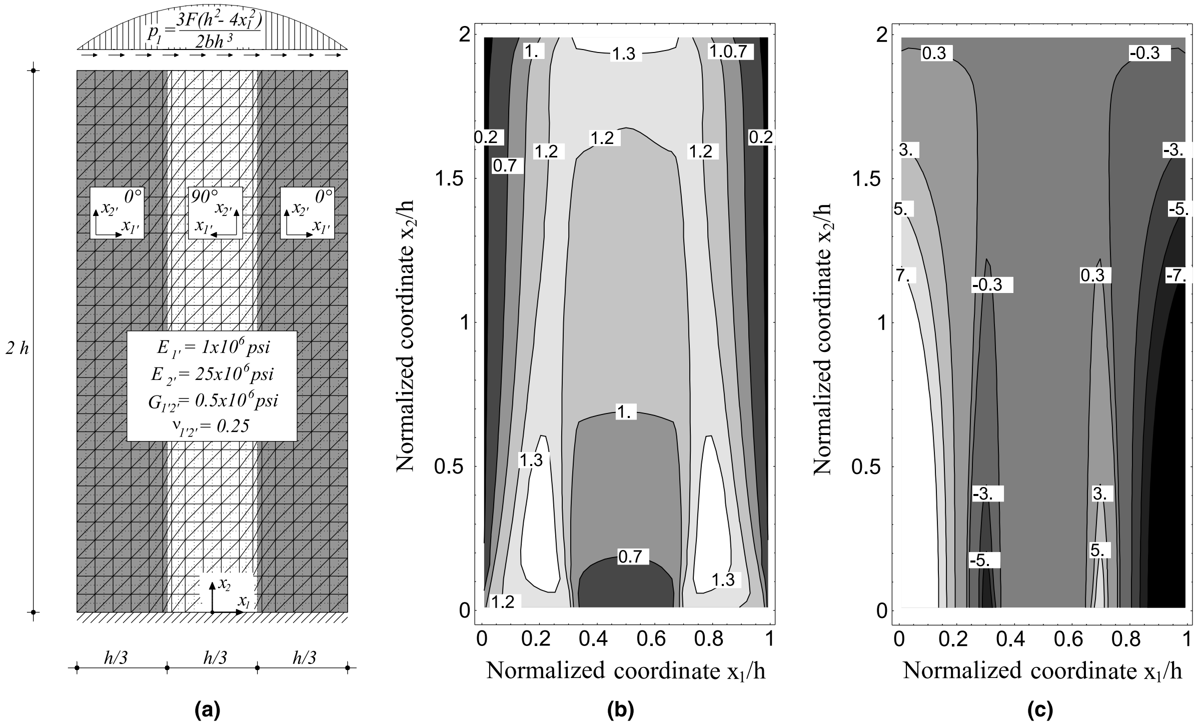

Fig. 2. Mesh and stress contours for a laminated beam: (a) LSM discretization; (b) bhT_{12}/F contours; (c) bhT_{22}/F contours.

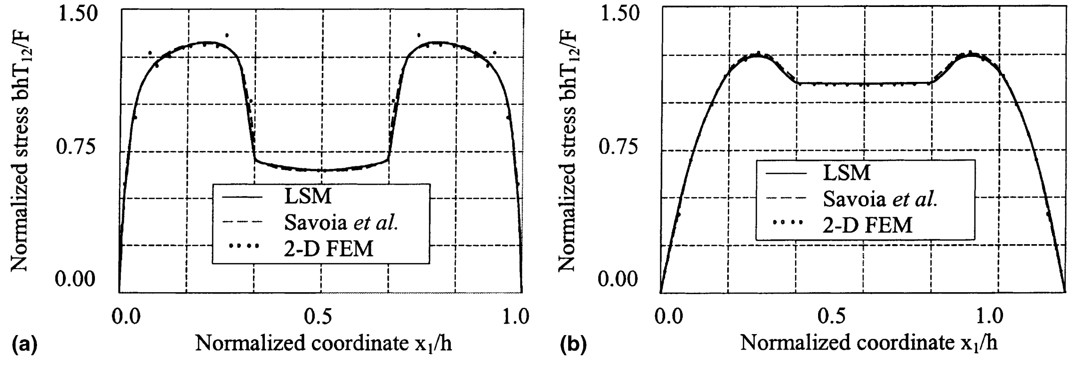

Fig. 3. Comparisons of LSM and other methods for tangential stresses in a thick laminated beam: (a) $x_2 = 0.125h$; (b) $x_2 = 0.9875h$.

value of $\bar{\mathbf{u}}$ at $\mathbf{x}_n \in \Gamma_u$, and $\mathbf{R}_n(\hat{\varphi})$ is the support reaction at the node $\mathbf{x}_n \in \Gamma_u$ corresponding to the forces $\hat{P}_n^s(\hat{\varphi})$. In Eq. (8),

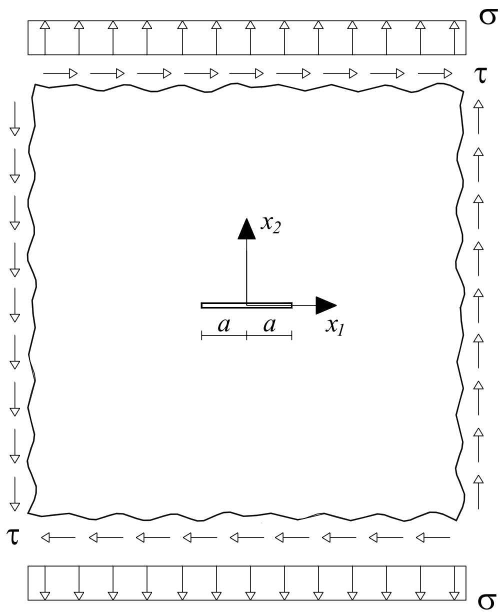

Fig. 4. Through crack in infinite orthotropic sheet.

$$\hat{\mathscr{A}}_n^{st} = \frac{\ell_n^s \ell_n^t \mathscr{A}[\hat{\mathbf{h}}_n^s \otimes \hat{\mathbf{h}}_n^s] \cdot \hat{\mathbf{h}}_n^t \otimes \hat{\mathbf{h}}_n^t}{4\ \mathrm{ar}(\hat{\Omega}_n)} = \frac{\ell_n^s \ell_n^t \mathscr{A}[\hat{\mathbf{k}}_n^s \otimes \hat{\mathbf{k}}_n^s] \cdot \hat{\mathbf{k}}_n^t \otimes \hat{\mathbf{k}}_n^t}{4\ \mathrm{ar}(\hat{\Omega}_n)}, \tag{9}$$

ℓ_n^s and ℓ_n^t being the lengths of the "bars" $n-s$ and $t-s$, respectively. The unit vector $\hat{\mathbf{k}}_n^s$ is in the direction of $\mathbf{x}_s - \mathbf{x}_n$.

Eq. (8) shows that the modified functional E_h can be interpreted as the complementary energy of a non-conventional pin-joint truss. It couples the contribution due to axial forces at each node.

Since every $\hat{\varphi} \in S_{\mathrm{ph}}$ is completely determined by its nodal values, it is a simple task to find the optimum conditions of E_h, that can be reduced to a linear system in the unknown $\hat{\varphi}_h = \{\hat{\varphi}_{h_1}, \hat{\varphi}_{h_2}, \ldots, \hat{\varphi}_{h_N}\}$. Such a system can be easily constructed [2] by using the summation and assembling operations. Numerical integrations are not necessary.

Let φ_o denote the minimizer of the original functional E in Eq. (3). In [1,2], the convergence of the LSM solutions $\hat{\varphi}_h$ to φ_o (as $h \to 0$) has been shown, under some regularity and uniformity assumptions about the meshes Π_h and $\hat{\Pi}_h$. Note that LSM can be embedded in the general theory of

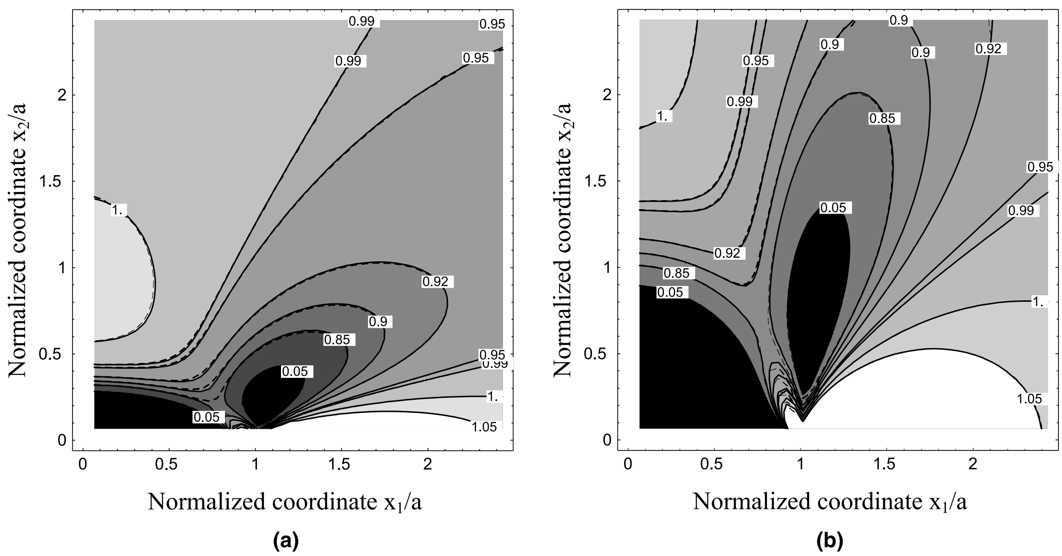

Fig. 5. Stress contours for orthotropic sheet with through crack in Mode II: (a) T_{12}/τ ($\beta_1 = 1, \beta_2^2 = 10$); (b) T_{12}/τ ($\beta_1 = 1, \beta_2^2 = 0.1$).

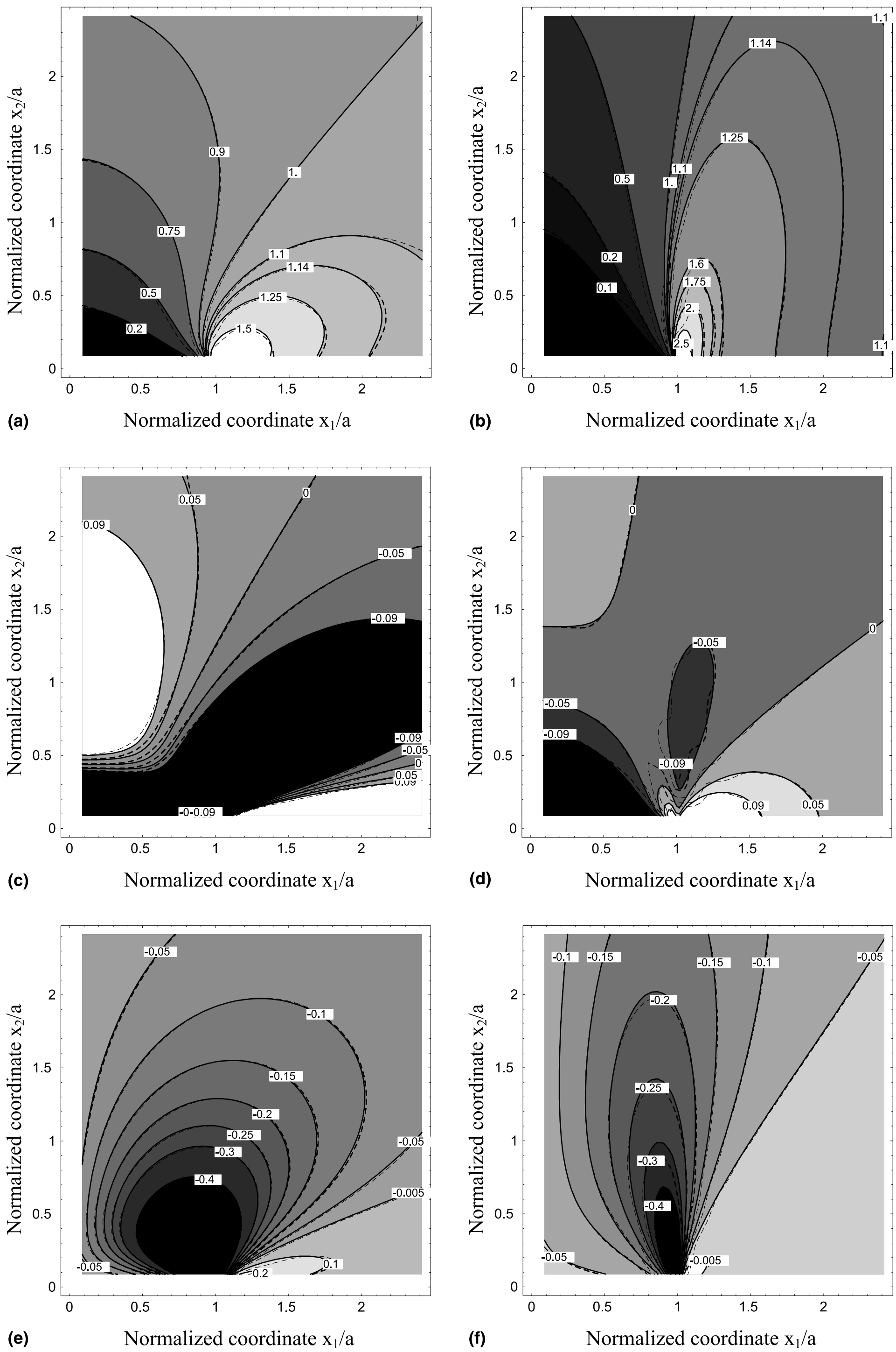

Fig. 6. Stress contours for orthotropic sheet with through crack in Mode I: (a) T_{22}/σ ($\beta_1 = 1, \beta_2^2 = 10$); (b) T_{22}/σ ($\beta_1 = 1, \beta_2^2 = 0.1$); (c) T_{11}/σ ($\beta_1 = 1, \beta_2^2 = 10$); (d) T_{11}/σ ($\beta_1 = 1, \beta_2^2 = 0.1$); (e) T_{12}/σ ($\beta_1 = 1, \beta_2^2 = 10$); (f) T_{12}/σ ($\beta_1 = 1, \beta_2^2 = 0.1$).

mixed finite element methods. In particular, it is possible to prove that the quantities

$$\mathbf{T}_n(\hat{\varphi}_h) = \frac{1}{\mathrm{ar}(\hat{\Omega}_n)} \sum_{s=1}^{S_n} \frac{\ell_n^s}{2} \hat{P}_n^s(\hat{\varphi}_h) \mathbf{k}_n^s \otimes \mathbf{k}_n^s$$
$$\forall n \in \{1, 2, \ldots, N\} \tag{10}$$

approximate the mean values of $\mathbf{T} - \mathbf{T}^*$ over the dual elements $\hat{\Omega}_n$. Thus, they can be interpolated to obtain an approximation of the actual stress distribution of the body.

3. Numerical results

Numerical results will be presented for two typical problems with stress concentrations and border effects.

3.1. Laminated beam

The first deals with a thick laminated beam clamped at one end and subjected to a parabolic distribution of tangential tractions at the other end. This distribution is equivalent to a shear force F. The composite material is characterized by a high ratio between the Young's modulus in the fiber direction $x_{2'}$ and that in the transverse direction $x_{1'}$, i.e., $E_{2'}/E_{1'} = 25$. The laminated stacking sequence is 0/90/0, while the length-to-thickness ratio ℓ/h is 2. For this example, high stress gradient prevails near the 0/90 interface and at the clamped end [5]. In Fig. 2, the LSM discretization is shown (558 DOF) together with the corresponding results for tangential and axial stresses ($b =$ beam width). In particular, the contour lines of T_{11} and T_{22} obtained by interpolation of the nodal values $T_{12}(\hat{\varphi}_h)$ and $T_{22}(\hat{\varphi}_h)$ are displayed.

In Fig. 3 a comparison between the LSM approximation for T_{12} and the numerical solution in [5] for two different values of the longitudinal coordinate x_2 are obtained. Note that $x_2 = 0.125h$ corresponds to a cross-section of the beam that is very close to the clamped end. Here, the stress distribution is strongly influenced by border effects.

3.2. Crack in orthotropic sheet

The second problem deals with a through crack in an infinite orthotropic sheet (Fig. 4). In this case, the material properties were expressed in terms of dimensionless parameters β_1 and β_2

$$\beta_1\beta_2 = \sqrt{E_1/E_2}, \qquad \beta_1 + \beta_2 = \sqrt{2(\sqrt{E_1/E_2} + E_1/2G_{12} - \nu_{12})}. \tag{11}$$

The problem was reduced to a domain of side five times larger than the crack length a by applying boundary forces corresponding to the traction of the analytical solution given in [6]. In particular, only a quarter of the domain was discretized, employing a 25×25 grid (676 DOF) for mode I loading, and a 50×50 grid (2601 DOF) for mode II loading.

Figs. 5 and 6 compare the contour lines of the Cartesian stress components obtained through the LSM (broken lines) with those descending from the analytical solution (solid lines). Two couples of values of β_1 and β_2 were considered. They are: $\beta_1 = 1$, $\beta_2^2 = 10$ ($x_1 =$ strong axis) and $\beta_1 = 1$, $\beta_2^2 = 0.1$ ($x_2 =$ strong axis).

4. Conclusion

It can be concluded that the lumped stress method is an unconstrained force method suited for treating plane elastic problems with stress singularities. This is accomplished by modeling the plane body through a non-conventional pin-joint truss structure. It requires a reduced number of degrees of freedom using only the summation and assembling operations. Generalization of the method to problems with stress constraints, such as those involving zero-tension or elasto-plastic materials, will be developed in future works.

References

[1] F. Fraternali, M. Angelillo, A. Fortunato, The lumped stress method and the discrete-continuum approximation – Part I: Theory Reports of the Department of Civil Engineering, University of Salerno, No. 79, 1999, 25 pp.

[2] F. Fraternali, M. Angelillo, The lumped stress method and the discrete-continuum approximation – Part II: Applications, Reports of the Department of Civil Engineering, University of Salerno, No. 80, 1999, 27 pp.
[3] R. Glowinski, Approximations Externes, par Eléments Finis de Lagrange d'Ordre Un et Deux, du Problème de Dirichlet pour l'Opérateur Biharmonique, Méthodes Itératives de Résolution des Problèmes Approchés, Topics in Numerical Analysis, Academic Press, London, 1973, pp. 123–171.
[4] C. Davini, I. Pitacco, Relaxed notions of curvature and a lumped strain method for elastic plates, SIAM J. Numer. Anal. 35 (1998) 677–691.
[5] M. Savoia, F. Laudiero, A. Tralli, A refined theory for laminated beams – Part II: An iterative variational approach, Meccanica 28 (1993) 217–225.
[6] G.C. Sih, E.P. Chen, in: Cracks in Composite Materials, Martinus Nijhoff, The Hague, 1981.